MATH 162M · Precalculus I

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1 THE WORDS, IN PLAIN ENGLISH

coefficient	the number in front of a letter: in $5x^2$ it is 5
constant term	the number with no x attached
degree	the highest power of x
polynomial	numbers times whole-number powers of x, added up
domain	every x you are allowed to put in
zero (root)	an x that makes $f(x) = 0$
conjugate	same two terms, middle sign flipped: $3 + \sqrt{2}$ and $3 - \sqrt{2}$
discriminant	$b^2 - 4ac$, the part under the root; its sign tells you how many real answers
extraneous	an answer the algebra makes that fails the original
asymptote	a line the graph gets closer and closer to far out; it can cross a horizontal one
multiplicity	how many times a factor repeats: $(x-2)^3$ is 3
one-to-one	no two inputs give the same output
logarithm	the exponent: $\log_2 8 = 3$ because $2^3 = 8$

2 ALGEBRAIC OPERATIONS

$a^m a^n = a^{m+n} \cdot \dfrac{a^m}{a^n} = a^{m-n} \cdot (a^m)^n = a^{mn}$

$(ab)^n = a^n b^n \cdot a^0 = 1 \cdot a^{-n} = \dfrac{1}{a^n}\ (a \neq 0)$

$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$ for $a \geq 0$

$\sqrt{ab} = \sqrt{a}\,\sqrt{b} \cdot \sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$ for $a \geq 0, b > 0$

e.g. Rationalize:

$$\frac{1}{3+\sqrt{2}} \cdot \frac{3-\sqrt{2}}{3-\sqrt{2}} = \frac{3-\sqrt{2}}{7}$$

Complex numbers $i = \sqrt{-1}, i^2 = -1 \cdot (a+bi)(a-bi) = a^2 + b^2$

⚠ $\sqrt{x^2} = |x|$, not x

FACTORING & SPECIAL PRODUCTS

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$a^2 - b^2 = (a+b)(a-b)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

→ Order: pull out the greatest common factor (GCF), then count terms. **2 terms:** difference of squares, or sum or difference of cubes. **3 terms:** find two numbers that multiply to ac and add to b. **4 terms:** grouping.

e.g. $x^2 + 5x + 6$: $2 \cdot 3 = 6$ and $2 + 3 = 5$, so $(x+2)(x+3)$

⚠ $a^2 + b^2$ does not factor over the real numbers

RATIONAL EXPRESSIONS

→ Factor everything first, then cancel common **factors**.

Multiply $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$ · Divide: flip the second fraction, then multiply

Add or subtract rewrite over the least common denominator (LCD), then combine the numerators

→ Complex fraction: multiply top and bottom by the LCD of all the small fractions.

Excluded values anything that makes a denominator 0, including the top of a fraction you divide by

e.g. $\dfrac{x^2-9}{x+3} = \dfrac{(x+3)(x-3)}{x+3} = x - 3$, with $x \neq -3$

⚠ Cancel factors, never terms: $\dfrac{x+3}{3} \neq x + 1$

3 EQUATIONS

Quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

e.g. $x^2 + 2x - 3 = 0$: $b^2 - 4ac = 16$, so $x = \dfrac{-2 \pm 4}{2} = 1$ or -3

Discriminant $b^2 - 4ac$: positive, two real · zero, one real · negative, two complex

Complete the square $x^2 + bx = \left(x + \dfrac{b}{2}\right)^2 - \left(\dfrac{b}{2}\right)^2$

$|X| = k$ with $k \geq 0$: $X = k$ or $X = -k$ · $k < 0$: no solution

→ Radical: isolate the root, raise both sides to the power, solve, **check every answer**.

→ Rational: multiply by the LCD, solve, throw out excluded values.

→ Quadratic type: $x^4 - 5x^2 + 4 = 0$, let $u = x^2$

INEQUALITIES

⚠ Multiply or divide by a negative: flip the sign.

e.g. $-2x < 6$: divide by -2 and flip, $x > -3$

$|X| < k$ means $-k < X < k$

$|X| > k$ means $X < -k$ or $X > k$

→ Polynomial or rational: get 0 on one side, find the zeros and the undefined points, test one number in each interval (sign chart).

Undefined points are never included; zeros only for $<$ or $>$

Interval notation [ ] includes, ( ) excludes, $\pm\infty$ always takes ( )

GRAPHS & LINES

Distance $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Midpoint $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Circle $(x-h)^2 + (y-k)^2 = r^2$, center (h, k)

Slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ · point-slope $y - y_1 = m(x - x_1)$

$y = mx + b$ · parallel: same m · perpendicular: $m_2 = -\dfrac{1}{m_1}$ (horizontal ⊥ vertical)

e.g. Slope through $(1, 2)$ and $(3, 8)$: $\dfrac{8-2}{3-1} = 3$

→ Intercepts: for the x-intercept set $y = 0$; for the y-intercept set $x = 0$.

Symmetry x-axis if $(x, -y)$ also works · y-axis if $(-x, y)$ · origin if $(-x, -y)$

FUNCTIONS

Domain, rule out a denominator of 0 · an even root of a negative · a log of 0 or less

Composition $(f \circ g)(x) = f(g(x))$, inside first

e.g. $f(x) = x^2, g(x) = x + 1$: $f(g(2)) = f(3) = 9$

Difference quotient $\dfrac{f(x+h) - f(x)}{h}$ (becomes the derivative in Calc 1)

Average rate of change on $[a, b]$: $\dfrac{f(b) - f(a)}{b - a}$

Even $f(-x) = f(x)$ · Odd: $f(-x) = -f(x)$

→ Inverse (only if f is one-to-one): swap x and y, solve for y. Check $f(f^{-1}(x)) = x$ both ways.

One-to-one passes the horizontal line test · f^{-1} is f mirrored over $y = x$

⚠ $f^{-1}(x) \neq \dfrac{1}{f(x)}$

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GRAPHS OF FUNCTIONS: TRANSFORMATIONS

$$y = a\,f\big(b(x-h)\big)+k$$

h: right $h\cdot$ k: up k

$|a|>1$ stretch · $0<|a|<1$ shrink · $a<0$ flip over the x-axis

b: horizontal scale by $\dfrac{1}{|b|}\cdot$ $b<0$ flip over the y-axis

e.g. Factor out b first: $f(2x+6)=f\big(2(x+3)\big)$ is a horizontal shrink by half and a shift left 3 (not 6).

⚠ $f(x+3)$ moves LEFT 3; $f(x)+3$ moves UP 3

POLYNOMIAL FUNCTIONS

Vertex form $f(x)=a(x-h)^2+k$, vertex (h,k)

From ax^2+bx+c: vertex at $x=-\dfrac{b}{2a}$, then $y=f\left(-\dfrac{b}{2a}\right)$

e.g. x^2-4x+1: $x=-\dfrac{-4}{2\cdot 1}=2$, $y=4-8+1=-3$, vertex $(2,-3)$

End behavior follows the leading term a_nx^n: even n, both ends go the way of the sign of a_n; odd n, the right end follows the sign of a_n and the left end goes the other way

Zero of odd multiplicity — the graph crosses · even multiplicity: touches and turns

Degree n: at most n real zeros and $n-1$ turning points

RATIONAL FUNCTIONS

Hole where a factor cancels completely from the bottom · vertical asymptote where the denominator is 0 after canceling

Horizontal asymptote, degree top n vs bottom m: $n<m, y=0$ · $n=m, y=\dfrac{\text{lead coef top}}{\text{lead coef bottom}}$ · $n>m$, none ($n=m+1$ gives a slant asymptote by long division)

e.g. $\dfrac{2x+1}{x-3}$: vertical $x=3$, horizontal $y=2$

THEORY OF EQUATIONS

Remainder theorem — $f(x)\div(x-c)$ leaves remainder $f(c)$

Factor theorem — $x-c$ is a factor exactly when $f(c)=0$

e.g. $f(x)=x^3-1$: $f(1)=0$, so $x-1$ is a factor

Rational zero candidates — $\pm\dfrac{p}{q}$, p divides the constant term, q divides the leading coefficient (integer coefficients)

→ Test candidates with synthetic division. Remainder 0 means a zero: keep going on the smaller quotient.

Descartes — positive zeros = sign changes in $f(x)$, or fewer by an even number; negative zeros: count in $f(-x)$

Fundamental theorem of algebra: degree n has exactly n complex zeros, counting multiplicity

Real coefficients — complex zeros come in pairs $a\pm bi$

SYSTEMS OF EQUATIONS

→ Substitution: solve one equation for a variable, plug it into the other.

→ Elimination: scale the equations so one variable cancels when you add.

e.g. $y=2x$ and $x+y=6$: $x+2x=6$, so $x=2, y=4$

One solution · none ($0=5$, parallel lines) · infinitely many ($0=0$, same line)

Row operations — swap rows · multiply a row by a nonzero number · add a multiple of one row to another

$$\det\begin{bmatrix}a & b\\ c & d\end{bmatrix}=ad-bc$$

Nonlinear system — substitute, then check each pair in both equations

EXPONENTIAL FUNCTIONS

$f(x)=a\cdot b^x$ with $a>0$: $b>1$ growth · $0<b<1$ decay · passes $(0,a)$ · asymptote $y=0$

Compound interest — $A=P\left(1+\dfrac{r}{n}\right)^{nt}$

Continuous — $A=Pe^{rt}\cdot e\approx 2.71828$

Growth or decay — $A=A_0e^{kt}$ ($k>0$ grows, $k<0$ decays)

e.g. Same base, set the exponents equal: $2^{x+1}=2^5$ gives $x=4$.

LOGARITHMIC FUNCTIONS

$y=\log_b x$ means $b^y=x$, for $b>0, b\neq 1$, $x>0$ (the log IS the exponent)

e.g. $\log_3 x=2$ means $x=3^2=9$

$\log_b 1=0\cdot\log_b b=1\cdot\log_b b^x=x\cdot b^{\log_b x}=x$ $(x>0)$

Product — $\log_b(MN)=\log_b M+\log_b N$ $(M,N>0)$

Quotient — $\log_b\dfrac{M}{N}=\log_b M-\log_b N$ ($M,N>0$)

Power — $\log_b M^p=p\log_b M$ for $M>0$

Change of base — $\log_b x=\dfrac{\ln x}{\ln b}\cdot$ ln is $\log_e$, log is $\log_{10}$

→ Exponential equation: isolate the power, take ln of both sides.

→ Log equation: condense to one log, rewrite as an exponential, **check x is in the domain**.

⚠ $\log(a+b)\neq\log a+\log b\cdot\dfrac{\log a}{\log b}\neq\log\dfrac{a}{b}$

WHICH MOVE?

If you see… → do this. First match wins.

- x^2 and no plain x term → **Isolate x^2, take $\pm\sqrt{}$**
- Quadratic → **Factor if it is quick; otherwise the formula**
- Degree 3 or more → **Rational zero candidates, synthetic division**
- x in a denominator → **Multiply by the LCD; check excluded values**
- x under a root → **Isolate the root, raise to the power, check**
- x in an exponent → **Same base: equal exponents; else take ln**
- x inside a log → **Condense, rewrite as exponential, check**
- Absolute value → **Split into two cases**
- Inequality with x^2 or a fraction → **Zeros and undefined points, sign chart**

⚠ COMMON SLIPS

Tick one when it bites you. Two ticks: slow down there.

- ☐ $(a+b)^2\neq a^2+b^2$ (multiply it out)
- ☐ $\sqrt{a^2+b^2}\neq a+b$
- ☐ $-3^2=-9$, but $(-3)^2=9$
- ☐ Distribute the minus: $-(x-4)=-x+4$
- ☐ Use $+$ when you square-root both sides

QUICK CHECKS

Cover the sheet. Do these cold. Answers are upside down.

1. Solve $x^2-6x+5=0$
2. Solve $|2x-1|<5$
3. Domain of $f(x)=\dfrac{\sqrt{x-3}}{x-5}$
4. Inverse of $f(x)=2x+7$
5. Solve $3^{x+1}=81$
6. Solve $\log_2 x+\log_2(x-2)=3$
7. Remainder of x^3-2x+4 divided by $x-2$
8. Solve $x+y=5$ and $x-y=1$

1. $x=1,\ 5$ **2.** $-2<x<3$ **3.** $[3,5)\cup(5,\infty)$ **4.** $f^{-1}(x)=\dfrac{x-7}{2}$ **5.** $x=3$ **6.** $x=4$ ($x=-2$ rejected) **7.** 8 **8.** $(3,\ 2)$