MATH 163 · Precalculus II

Start here 1 The words, in plain English 2 Trigonometric functions: unit circle 3 Trigonometric identities

Study tool. Not for use on a test unless your instructor allows it.

1 THE WORDS, IN PLAIN ENGLISH

radian	angle measured by arc length on a radius-1 circle: π rad $= 180^\circ$
coterminal	angles that end on the same ray: 30° and 390°
reference angle	the acute angle between the angle's ending side and the x-axis
quadrant	one of the four quarters of the plane, I to IV counterclockwise
amplitude	half the height from lowest to highest point
period	how far along x before the graph repeats
phase shift	how far the graph slides left or right
midline	the horizontal line halfway between top and bottom
identity	an equation true for every allowed x
arcsine	the angle in $[-\pi/2, \pi/2]$ whose sine is given: $\sin^{-1}(1/2) = \pi/6$
magnitude	a vector's length
orthogonal	at right angles; for vectors, dot product 0
modulus, argument	a complex number's distance from 0, and its angle
focus	a fixed point (two for an ellipse or hyperbola) the conic is built around

EXPONENTIAL & LOGARITHMIC FUNCTIONS (FROM 162M)

$y = \log_b x$ means $b^y = x$ ($b > 0, b \neq 1, x > 0$) · ln is $\log_e$

$\log_b(MN) = \log_b M + \log_b N$ · $\log_b \dfrac{M}{N} = \log_b M - \log_b N$ $(M, N > 0)$

→ Exponential equation: isolate, take ln. Log equation: condense, rewrite as exponential, check the domain.

e.g. $e^{2x} = 5$: $2x = \ln 5$, so $x = \dfrac{\ln 5}{2} \approx 0.805$

ANGLES

π rad $= 180^\circ$ · degrees to radians $\times \dfrac{\pi}{180}$ · radians to degrees $\times \dfrac{180}{\pi}$

Arc length $s = r\theta$ · sector area $A = \dfrac{1}{2}r^2\theta$ (θ in radians)

e.g. $r = 4, \theta = 150^\circ = \dfrac{5\pi}{6}$: $s = 4 \cdot \dfrac{5\pi}{6} = \dfrac{10\pi}{3}$

Linear speed $v = r\omega$ (ω in rad/time) · coterminal: add or subtract 2π (360°)

Reference angle (acute, to the x-axis; first bring θ into $[0, 2\pi)$; QI–QIV are quadrants I–IV, counterclockwise from upper right): QII $\pi - \theta$ · QIII $\theta - \pi$ · QIV $2\pi - \theta$

TRIGONOMETRIC FUNCTIONS: RIGHT TRIANGLES

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$ · $\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$ · $\tan\theta = \dfrac{\text{opp}}{\text{adj}}$ (SOH CAH TOA)

$\csc\theta = \dfrac{1}{\sin\theta}$ · $\sec\theta = \dfrac{1}{\cos\theta}$ · $\cot\theta = \dfrac{1}{\tan\theta}$

e.g. Opposite 3, hypotenuse 5: adjacent 4, so $\sin\theta = \dfrac{3}{5}, \cos\theta = \dfrac{4}{5}, \tan\theta = \dfrac{3}{4}$

30-60-90 sides $1 : \sqrt{3} : 2$ (opposite $30^\circ, 60^\circ, 90^\circ$) · 45-45-90 sides $1 : 1 : \sqrt{2}$

2 TRIGONOMETRIC FUNCTIONS: UNIT CIRCLE

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
	0°	30°	45°	60°	90°
$\sin$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef

A point on the unit circle is $(\cos\theta, \sin\theta)$ · $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Positive QI all · QII sin · QIII tan · QIV cos, and their reciprocals ("All Students Take Calculus")

→ Exact value anywhere: reference angle from the table, sign from the quadrant.

e.g. $\sin\dfrac{5\pi}{6}$: reference angle $\dfrac{\pi}{6}$, QII so positive, $\dfrac{1}{2}$

GRAPHS OF TRIGONOMETRIC FUNCTIONS

$y = A\sin\big(B(x - C)\big) + D$

Amplitude $|A|$ · period $\dfrac{2\pi}{|B|}$ · phase shift C · midline $y = D$

e.g. $y = 2\sin 3x + 1$: amplitude 2, period $\dfrac{2\pi}{3}$, midline $y = 1$

tan and cot: period $\dfrac{\pi}{|B|}$ · $y = \tan x$ has asymptotes at $x = \dfrac{\pi}{2} + k\pi$

sin, cos: range $[-1, 1]$ · domain all real numbers

→ Sketch one period in four equal steps: five key points.

⚠ Factor out B: $\sin(2x - \pi) = \sin\big(2(x - \dfrac{\pi}{2})\big)$, the shift is $\dfrac{\pi}{2}$, not π

INVERSE TRIGONOMETRIC FUNCTIONS

$\sin^{-1} x$ returns an angle in $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$\cos^{-1} x$ returns $[0, \pi]$ · $\tan^{-1} x$ returns $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

e.g. $\cos^{-1}\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$, the angle in $[0, \pi]$

⚠ $\sin^{-1} x$ means arcsine, not $\dfrac{1}{\sin x}$

3 TRIGONOMETRIC IDENTITIES

$\sin^2 x + \cos^2 x = 1$ · $1 + \tan^2 x = \sec^2 x$ · $1 + \cot^2 x = \csc^2 x$

$\cos(-x) = \cos x$ · $\sin(-x) = -\sin x$ · $\tan(-x) = -\tan x$

Cofunction $\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$

$\sin(a \pm b) = \sin a\cos b \pm \cos a \sin b$

$\cos(a \pm b) = \cos a\cos b \mp \sin a \sin b$

e.g. $\sin 75^\circ = \sin(45^\circ + 30^\circ) = \dfrac{\sqrt{6} + \sqrt{2}}{4}$

$\tan(a \pm b) = \dfrac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$

$\sin 2x = 2\sin x\cos x$ · $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

$\cos 2x = \cos^2 x - \sin^2 x$, also $2\cos^2 x - 1$ or $1 - 2\sin^2 x$

Power-reducing $\sin^2 x = \dfrac{1 - \cos 2x}{2}$ · $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

Half-angle $\sin\dfrac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{2}}$ · $\cos\dfrac{x}{2} = \pm\sqrt{\dfrac{1 + \cos x}{2}}$, sign from the quadrant of $\dfrac{x}{2}$

→ Proving one: work on the messier side only, rewrite in sin and cos, combine fractions, factor, look for $\sin^2 + \cos^2$.

LAW OF SINES & LAW OF COSINES

Given parts, in order around the triangle: S = side, A = angle. SAS = two sides and the angle between them; SSA = two sides and an angle not between them.

Sines $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, for AAS, ASA, SSA

e.g. $A = 30^\circ, B = 45^\circ, a = 10$: $b = \dfrac{10 \sin 45^\circ}{\sin 30^\circ} = 10\sqrt{2}$

Cosines $c^2 = a^2 + b^2 - 2ab\cos C$, for SAS, SSS

Area $= \dfrac{1}{2}ab\sin C$ · Heron: $\sqrt{s(s-a)(s-b)(s-c)}$, $s = \dfrac{a+b+c}{2}$

→ SSS: find the largest angle first; the other two are then acute.

⚠ SSA can give 0, 1, or 2 triangles: if B works, also try $180^\circ - B$; keep it only if $A + (180^\circ - B) < 180^\circ$.

VECTORS

$\mathbf{v} = \langle a, b\rangle \cdot |\mathbf{v}| = \sqrt{a^2+b^2}$ · unit vector $\dfrac{\mathbf{v}}{|\mathbf{v}|}$

From length and angle $\mathbf{v} = \langle |\mathbf{v}|\cos\theta,\ |\mathbf{v}|\sin\theta\rangle$

Direction $\tan\theta = \dfrac{b}{a}$, then fix the quadrant

Dot product $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2$ · $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$

Perpendicular (orthogonal) exactly when $\mathbf{u}\cdot\mathbf{v} = 0$

e.g. $\langle 1, 2\rangle \cdot \langle 4, -2\rangle = 4 - 4 = 0$, so they are orthogonal

POLAR FORM OF COMPLEX NUMBERS

$z = a + bi = r(\cos\theta + i\sin\theta) \cdot r = \sqrt{a^2+b^2}$ · $\tan\theta = \dfrac{b}{a}$ (check the quadrant)

e.g. $1 + i$: $r = \sqrt{2}, \theta = \dfrac{\pi}{4}$, so $\sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

Multiply multiply the r's, add the angles · Divide: divide the r's, subtract the angles

De Moivre $z^n = r^n(\cos n\theta + i\sin n\theta)$

nth roots: $\sqrt[n]{r}\left(\cos\dfrac{\theta+2k\pi}{n} + i\sin\dfrac{\theta+2k\pi}{n}\right)$, $k = 0, 1, \ldots, n-1$

Polar points $x = r\cos\theta \cdot y = r\sin\theta$ · $r^2 = x^2 + y^2$

BINOMIAL THEOREM

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

$\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ = entry k of row n in Pascal's triangle (count both from 0)

e.g. $\binom{6}{2} = \dfrac{6!}{2!\,4!} = 15$

The term with b^k (term number $k+1$): $\binom{n}{k}a^{n-k}b^k$

⚠ Keep the sign inside b: in $(x-2)^5$, $b = -2$.

TRIGONOMETRIC EQUATIONS

→ **1** Isolate the trig function. **2** Find the reference angle. **3** List every angle in $[0, 2\pi)$ with the right sign. **4** General solution: add $2k\pi$ (for tan and cot, $k\pi$).

→ Two different trig functions: use an identity to get one, or factor.

e.g. $\sin 2x = \dfrac{\sqrt{2}}{2}$ on $[0, 2\pi)$: $2x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{9\pi}{4}, \dfrac{11\pi}{4}$ on $[0, 4\pi)$, so $x = \dfrac{\pi}{8}, \dfrac{3\pi}{8}, \dfrac{9\pi}{8}, \dfrac{11\pi}{8}$

⚠ Never divide both sides by $\sin x$: factor it out, or you lose the $\sin x = 0$ answers.

CONIC SECTIONS

Parabola $(x-h)^2 = 4p(y-k)$: vertex (h, k), focus $(h, k+p)$, opens up if $p > 0$ · $(y-k)^2 = 4p(x-h)$ opens right if $p > 0$

Ellipse $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, long axis horizontal; for vertical, a^2 goes under $(y-k)^2$. Always $a > b$ and $c^2 = a^2 - b^2$

Hyperbola $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$, $c^2 = a^2 + b^2$, asymptotes $y - k = \pm\dfrac{b}{a}(x-h)$; $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$ opens up and down, slopes $\pm\dfrac{a}{b}$ (a^2 always under the positive term)

e.g. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$: $a = 5, b = 3$, $c = \sqrt{25-9} = 4$, foci $(\pm 4, 0)$

→ Identify $Ax^2 + Cy^2 + Dx + Ey + F = 0$: $A = C \neq 0$ circle · $AC > 0$ ellipse · $AC < 0$ hyperbola · one squared term parabola. Then complete the square: a 0 or negative on the right can mean a point, two lines, or no graph.

WHICH MOVE?

If you see… → do this. First match wins.

- Right triangle → **SOH CAH TOA**
- SSS or SAS → **Law of Cosines**
- AAS or ASA → **Law of Sines**
- SSA → **Law of Sines, then look for a second triangle**
- Only $2x$ inside the trig function → **Solve for $2x$ on the doubled interval, then divide**
- $2x$ mixed with x, or two trig functions → **Double-angle identity, or factor**
- Power or root of a complex number → **Polar form, De Moivre**
- One term of $(a+b)^n$ → **Term formula, not the full expansion**

⚠ COMMON SLIPS

Tick one when it bites you. Two ticks: slow down there.

- ☐ $\sin 2x \neq 2\sin x$ and $\sin(a+b) \neq \sin a + \sin b$
- ☐ Degree mode or radian mode: match the problem
- ☐ Arc length and sector area need radians
- ☐ Find every solution in the interval, not just the first
- ☐ After $\tan^{-1}$, check the quadrant

QUICK CHECKS

Cover the sheet. Do these cold. Answers are upside down.

1. $\cos\dfrac{2\pi}{3}$
2. Solve $2\sin x - 1 = 0$ on $[0, 2\pi)$
3. Period of $y = 3\cos 4x$
4. $|\langle 3, -4\rangle|$
5. $(1+i)^8$
6. Coefficient of x^2 in $(x+2)^5$
7. $a = 7,\ b = 8,\ C = 60^\circ$. Find c.
8. Which conic is $x^2 + 4y^2 = 16$?

1. $-\dfrac{1}{2}$ 2. $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ 3. $\dfrac{\pi}{2}$ 4. 5 5. 16 6. 80

7. $c = \sqrt{57}$ 8. ellipse, $a = 4,\ b = 2$