MATH 211 · Calculus I

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1 THE WORDS, IN PLAIN ENGLISH

limit	the value $f(x)$ heads toward as x gets close to a
continuous	no break, hole or jump: $\lim_{x\to a} f(x) = f(a)$
indeterminate form	a form like $0/0$ or ∞/∞ (also $0\cdot\infty, \infty-\infty, 1^\infty, 0^0, \infty^0$): not an answer, a sign to do more work
derivative	the slope of the graph at one point; the instant rate of change
differentiable	has a derivative there: smooth, no corner, no break, no vertical tangent
tangent line	the line through the point with the same slope as the graph there
implicit	y is mixed into the equation, not solved for
critical number	an x in the domain where $f'=0$ or f' does not exist
extrema	highest and lowest values (maximum and minimum)
concave up	bends upward like a cup; $f''>0$ is the usual test
inflection point	where the bend changes direction
antiderivative	a function whose derivative is the one you have
definite integral	area between the graph and the x-axis from a to b; area below the axis counts negative
integrand	the function inside the integral sign

PRECALC YOU WILL USE EVERY DAY

$\sqrt{x} = x^{1/2}\cdot\dfrac{1}{x^n} = x^{-n}\cdot\dfrac{1}{\sqrt{x}} = x^{-1/2}$ (rewrite before differentiating)

e.g. $\dfrac{1}{\sqrt{x}} = x^{-1/2}$, so its derivative is $-\dfrac{1}{2}x^{-3/2}$

→ Most Calc 1 mistakes are algebra mistakes. Keep the 162M and 163 sheets.

2 LIMITS

$\lim_{x\to c} f(x)$ exists only when the left and right limits exist and are equal

→ **1** Plug in. A number, and f is continuous there? Done. At a piecewise break, take one-sided limits.

→ **2** $\dfrac{\text{nonzero}}{0}$: vertical asymptote. Check each side for $+\infty$ or $-\infty$.

→ **3** $\dfrac{0}{0}$: factor and cancel, multiply by the conjugate, or clear small fractions. Or L'Hôpital.

e.g. $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2}(x+2) = 4$

$\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1\cdot\lim\limits_{x\to 0}\dfrac{1-\cos x}{x} = 0$

$x\to+\infty$: divide top and bottom by the highest power of x in the denominator

Squeeze $g\le f\le h$ and $\lim g = \lim h = L$, so $\lim f = L$

CONTINUITY

Continuous at c: $f(c)$ is defined, $\lim_{x\to c} f(x)$ exists, and the two are equal

e.g. $\dfrac{x^2-1}{x-1}$ is not continuous at $x=1$: it is undefined there

Intermediate Value Theorem: continuous on $[a,b]$ means f takes every value between $f(a)$ and $f(b)$

THE DERIVATIVE

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

e.g. $f(x)=x^2$: $\dfrac{(x+h)^2-x^2}{h} = 2x+h\to 2x$

Tangent line at $x=a$: $y-f(a) = f'(a)(x-a)$

f' is the slope of the tangent, the instantaneous rate of change

Not differentiable at a corner, a cusp, a vertical tangent, or a break

3 DIFFERENTIATION RULES

$(c)' = 0\cdot(x^n)' = nx^{n-1}\cdot(cf)' = cf'\cdot(f\pm g)' = f'\pm g'$

Product $(fg)' = f'g + fg'$

Quotient $\left(\dfrac{f}{g}\right)' = \dfrac{f'g-fg'}{g^2}$

Chain $\big[f(g(x))\big]' = f'(g(x))\cdot g'(x)$

→ Chain rule in words: derivative of the outside (inside left alone), times the derivative of the inside.

e.g. $\big[(x^2+1)^5\big]' = 5(x^2+1)^4\cdot 2x$

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS

$(\sin x)' = \cos x\cdot(\cos x)' = -\sin x\cdot(\tan x)' = \sec^2 x$

$(\cot x)' = -\csc^2 x\cdot(\sec x)' = \sec x\tan x\cdot(\csc x)' = -\csc x\cot x$

$(e^x)' = e^x\cdot(a^x)' = a^x\ln a\ (a>0)$

$(\ln x)' = \dfrac{1}{x}\cdot(\log_a x)' = \dfrac{1}{x\ln a}\ (a>0,\ a\ne 1)$

$(\sin^{-1}x)' = \dfrac{1}{\sqrt{1-x^2}}\cdot$

$(\tan^{-1}x)' = \dfrac{1}{1+x^2}$

e.g. $(e^x\sin x)' = e^x\sin x + e^x\cos x$ (product rule)

→ The "co" functions (cos, cot, csc) carry the minus sign.

IMPLICIT DIFFERENTIATION

→ Differentiate both sides in x. Every time you differentiate a y, attach y'. Then solve for y'

e.g. $x^2+y^2=25$ gives $2x+2yy'=0$, so $y' = -\dfrac{x}{y}$

Variable in base and exponent (x^x): take ln of both sides first

APPLICATIONS: RELATED RATES & LINEAR APPROXIMATION

→ **1** Draw and label. **2** Write an equation linking the quantities. **3** Differentiate with respect to t. **4** THEN plug in the numbers.

e.g. $A=\pi r^2$ gives $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$. With $r=3$ and $\dfrac{dr}{dt}=2$, the area grows at 12π

Linear approximation $L(x) = f(a)+f'(a)(x-a)$ · differential $dy = f'(x)\,dx$

APPLICATIONS: EXTREMA & SHAPE

Critical number — c inside the domain with $f'(c) = 0$ or $f'(c)$ undefined

$f' > 0$ increasing · $f' < 0$ decreasing · at a critical number where f is continuous, f' from $+$ to $-$ is a local max, $-$ to $+$ a local min

$f'' > 0$ concave up · $f'' < 0$ concave down · inflection point: f continuous there and concavity changes

e.g. $f(x) = x^2 - 4x$: $f' = 2x - 4 = 0$ at $x = 2$, $f'' = 2 > 0$, local min $f(2) = -4$

Second derivative test — $f'(c) = 0$ with $f''(c) < 0$ is a local max; with $f''(c) > 0$ a local min; $f''(c) = 0$ tells you nothing, use the f' sign chart

→ Closed interval $[a, b]$, f continuous: evaluate f at every critical number AND both endpoints. Largest is the absolute max, smallest the absolute min.

Mean Value Theorem — if f is continuous on $[a, b]$ and differentiable on (a, b), then $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some c in (a, b)

APPLICATIONS: OPTIMIZATION

→ **1** Draw. **2** Write the quantity to maximize or minimize. **3** Use the constraint to get one variable. **4** State the domain. **5** Find the critical numbers: $f' = 0$ or f' undefined. **6** Compare them with the endpoints, or use the sign of f'. **7** Answer the question asked, with units.

e.g. Two numbers add to 10; largest product: $P = x(10 - x)$, $P' = 10 - 2x = 0, x = 5, P = 25$

L'HÔPITAL'S RULE

$\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$: $\lim \dfrac{f(x)}{g(x)} = \lim \dfrac{f'(x)}{g'(x)}$ if f and g are differentiable near the point, $g' \neq 0$ there, and the new limit exists or is $\pm\infty$

⚠ Differentiate top and bottom separately, not with the quotient rule.

e.g. $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = \lim\limits_{x \to 0} \dfrac{e^x}{1} = 1$

$0 \cdot \infty$ or $\infty - \infty$: rewrite as one fraction first

$1^\infty, 0^0, \infty^0$: take $\ln$, find that limit, then raise e to it

INTEGRATION: ANTIDERIVATIVES

$\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C$ for $n \neq -1$

$\int \dfrac{1}{x} \, dx = \ln|x| + C$ · $\int e^x \, dx = e^x + C$

e.g. $\int \left(3x^2 + \dfrac{1}{x}\right) dx = x^3 + \ln|x| + C$

$\int \cos x \, dx = \sin x + C$ ·
$\int \sin x \, dx = -\cos x + C$ ·
$\int \sec^2 x \, dx = \tan x + C$ ·
$\int \sec x \tan x \, dx = \sec x + C$ ·
$\int \dfrac{dx}{1 + x^2} = \tan^{-1} x + C$ ·
$\int \dfrac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C$

$\int a^x \, dx = \dfrac{a^x}{\ln a} + C$ for $a > 0, a \neq 1$

INTEGRATION: U-SUBSTITUTION

→ Pick u = the inside function whose derivative (up to a constant) is also there. Write $du = u'(x) \, dx$.

e.g. $\int xe^{x^2} \, dx$: $u = x^2$, $du = 2x \, dx$, answer $\dfrac{1}{2}e^{x^2} + C$

⚠ Definite integral: change the bounds to u-values, or switch back to x before plugging in.

INTEGRATION: DEFINITE INTEGRALS & THE FTC

Riemann sum — $\sum f(x_i^*) \, \Delta x$ with $\Delta x = \dfrac{b - a}{n}$

Fundamental Theorem of Calculus (FTC), part 1: f continuous, $\dfrac{d}{dx} \int_a^x f(t) \, dt = f(x)$

With a chain:

$$\frac{d}{dx} \int_a^{g(x)} f(t) \, dt = f(g(x)) \, g'(x)$$

FTC part 2 — f continuous on $[a, b]$, $F' = f$: $\int_a^b f(x) \, dx = F(b) - F(a)$

e.g. $\int_1^3 2x \, dx = 3^2 - 1^2 = 8$

$\int_a^a f = 0$ · $\int_a^b f = -\int_a^b f$

Average value — $\dfrac{1}{b - a} \int_a^b f(x) \, dx$ · area between curves $\int_a^b (\text{top} - \text{bottom}) \, dx$, split where the curves cross

WHICH MOVE?

If you see… → do this. First match wins.

- Any limit → **Plug in first**
- $\dfrac{0}{0}$ → **Factor, conjugate, or L'Hôpital**
- Product of two functions → **Product rule**
- A fraction → **Quotient rule, or rewrite with a negative exponent**
- Function inside a function → **Chain rule**
- x and y mixed together → **Implicit differentiation**
- "How fast is it changing" → **Related rates**
- "Largest, smallest, most" → **Optimization**
- Integral on the table → **Reverse the table, add C**
- An inside function and its derivative → **u-substitution**
- $\dfrac{d}{dx} \int_a^{g(x)}$ → **FTC part 1 and the chain rule**

⚠ COMMON SLIPS

Tick one when it bites you. Two ticks: slow down there.

- ☐ Missing the inner derivative in the chain rule
- ☐ Quotient rule: $f'g - fg'$ on top (order matters), g^2 below
- ☐ Forgetting $+C$
- ☐ $\int \dfrac{1}{x} \, dx = \ln|x| + C$, not the power rule
- ☐ $\dfrac{d}{dx}(e^2) = 0$, it is a constant

QUICK CHECKS

Cover the sheet. Do these cold. Answers are upside down.

1. $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$
2. $\dfrac{d}{dx}\left[x^2 \sin x\right]$
3. $\dfrac{d}{dx}\left[\ln(x^2 + 1)\right]$
4. $\dfrac{d}{dx}\left[e^{3x}\right]$
5. $\int_0^2 3x^2 \, dx$
6. $\int 2x \cos(x^2) \, dx$
7. $\lim\limits_{x \to \infty} \dfrac{3x^2 + 1}{5x^2 - x}$
8. Critical numbers of $f(x) = x^3 - 3x$

8. $x = \mp 1$

4. $3e^{3x}$ 5. 8 6. $\sin(x^2) + C$ 7. $\dfrac{3}{5}$

1. 6 2. $2x \sin x + x^2 \cos x$ 3. $\dfrac{2x}{x^2 + 1}$