MATH 212 · Calculus II

Start here 1 The words, in plain English 2 Integration by parts 3 Infinite series: tests

Study tool. Not for use on a test unless your instructor allows it.

1 THE WORDS, IN PLAIN ENGLISH

integrand	the function inside the integral sign
irreducible	a quadratic that will not factor over the reals: x^2+1
improper integral	an infinite bound, or a blow-up inside: done as a limit
converges	settles on one finite number
diverges	does not settle: grows without bound or keeps bouncing
sequence, series	a list $a_1, a_2, \ldots$; a series adds the list up
partial sum	S_n, the sum of the first n terms
absolutely convergent	still converges with every term made positive
conditionally convergent	converges, but not with every term made positive
radius of convergence	how far from the center it converges; test each endpoint alone
Taylor series	a function rebuilt as an infinite polynomial around $x=a$
parametric	x and y each given by a third variable t
polar	a point given by distance r from the origin and angle θ
normal vector	a vector at right angles to a plane

TRIGONOMETRIC SUBSTITUTION

$\sqrt{a^2-x^2}$: $x=a\sin\theta$, root $=a\cos\theta$

$\sqrt{a^2+x^2}$: $x=a\tan\theta$, root $=a\sec\theta$

$\sqrt{x^2-a^2}$: $x=a\sec\theta$, root $=a|\tan\theta|$, which is $a\tan\theta$ when $x\geq a$

→ Back to x: draw the right triangle the substitution describes.

e.g. $\displaystyle\int\frac{dx}{\sqrt{4-x^2}}$: $x=2\sin\theta$, $dx=2\cos\theta\,d\theta$, root $=2\cos\theta$, so $\displaystyle\int d\theta=\sin^{-1}\frac{x}{2}+C$

⚠ $\sqrt{\cos^2\theta}=|\cos\theta|$: drop the bars only once you show it is positive

2 INTEGRATION BY PARTS

$$\int u\,dv=uv-\int v\,du$$

→ Pick u by LIATE: Log, Inverse trig, Algebraic, Trig, Exponential. First on the list is u.

e.g. $\int x\cos x\,dx$: $u=x, v=\sin x$, so $x\sin x+\cos x+C$

Definite:

$$\int_a^b u\,dv=\big[uv\big]_a^b-\int_a^b v\,du$$

→ Cycle ($e^{ax}\sin bx$ or $e^{ax}\cos bx$): parts twice, same kind of u both rounds. The integral I comes back; move it left and solve for I.

$$\int e^{ax}\sin bx\,dx=\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}+C$$

TRIGONOMETRIC INTEGRALS

Odd power of sin: save one $\sin x$, rest to $1-\cos^2 x$, $u=\cos x$ ($du=-\sin x\,dx$)

e.g. $\int\sin^3 x\,dx$ with $u=\cos x$: $-\cos x+\dfrac{\cos^3 x}{3}+C$

Odd power of cos: save one $\cos x$, rest to $1-\sin^2 x$, $u=\sin x$

Both even $\quad\sin^2 x=\dfrac{1-\cos 2x}{2}\cdot\quad\cos^2 x=\dfrac{1+\cos 2x}{2}$

$\int\tan x\,dx=\ln|\sec x|+C\cdot$

$\int\sec x\,dx=\ln|\sec x+\tan x|+C$

$\int\csc x\,dx=\ln|\csc x-\cot x|+C$

PARTIAL FRACTIONS

→ Degree check first: top degree $\geq$ bottom degree means long division before splitting.

Distinct linear ($a\neq b$):

$$\frac{px+q}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}$$

Repeated $\quad\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}\cdot$

irreducible quadratic: $\dfrac{Ax+B}{x^2+bx+c}$

→ Cover-up: clear denominators, set $x=a$, then $x=b$; one constant drops out each time.

e.g. $\dfrac{1}{x(x+1)}=\dfrac{A}{x}+\dfrac{B}{x+1}$: $x=0$ gives $A=1$, $x=-1$ gives $B=-1$

$\displaystyle\int\frac{dx}{ax+b}=\frac{1}{a}\ln|ax+b|+C$, not the power rule

IMPROPER INTEGRALS

Infinite bound $\quad\displaystyle\int_a^\infty f\,dx=\lim_{t\to\infty}\int_a^t f\,dx$

Blow-up at b: $\displaystyle\lim_{t\to b^-}\int_a^t f\,dx\cdot$ at a: $\displaystyle\lim_{t\to a^+}\int_t^b f\,dx$

Blow-up inside, at c: split at c; it converges only if both pieces converge

e.g. $\displaystyle\int_0^1\frac{dx}{\sqrt{x}}=\lim_{t\to0^+}\big[2\sqrt{x}\big]_t^1=2$

p-tests $\quad\displaystyle\int_1^\infty\frac{dx}{x^p}$ converges iff $p>1\cdot$

$\displaystyle\int_0^1\frac{dx}{x^p}$ converges iff $p<1$

⚠ Scan for hidden blow-ups ($\ln x$ at 0, a zero denominator) before integrating.

POLAR COORDINATES

$x=r\cos\theta\cdot y=r\sin\theta\cdot r^2=x^2+y^2\cdot\tan\theta=\dfrac{y}{x}$

e.g. $r=2,\theta=\dfrac{\pi}{3}$: $x=2\cos\dfrac{\pi}{3}=1$, $y=2\sin\dfrac{\pi}{3}=\sqrt{3}$

Area $\quad A=\dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2\,d\theta\cdot$ between curves $\dfrac{1}{2}\displaystyle\int_\alpha^\beta\left(r_{\text{out}}^2-r_{\text{in}}^2\right)d\theta$

Arc length $\quad L=\displaystyle\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

→ Bounds trace the region once: $r=a\cos\theta$ closes on $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$; 0 to 2π counts it twice.

→ Intersections: set the r's equal, then check the pole separately.

Limaçon $r=a+b\sin\theta$ (or $\cos\theta$), $a,b>0$: inner loop when $a<b$; the loop lies between the θ's where $r=0$

⚠ Negative r plots opposite θ.

PARAMETRIC CURVES

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}\cdot\quad\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\Big/\frac{dx}{dt}$$

e.g. $x=t^2, y=t^3$: $\dfrac{dy}{dx}=\dfrac{3t^2}{2t}=\dfrac{3t}{2}$ $(t\neq0)$

Horizontal tangent $\quad\dfrac{dy}{dt}=0$ and $\dfrac{dx}{dt}\neq0\cdot$ vertical: $\dfrac{dx}{dt}=0$ and $\dfrac{dy}{dt}\neq0$

Arc length $L=\displaystyle\int_\alpha^\beta ds$, where

$$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Surface area $\quad S=2\pi\displaystyle\int_\alpha^\beta(\text{radius})\,ds$; radius is y about the x-axis, x about the y-axis (radius >0, curve traced once)

3 INFINITE SERIES: TESTS

Divergence $\lim a_n \neq 0$ means $\sum a_n$ diverges

⚠ $a_n \to 0$ alone proves nothing.

e.g. $\sum \frac{n}{2n+1}$: the terms go to $\frac{1}{2} \neq 0$, so it diverges

Geometric $\sum_{n=0}^{\infty} ar^n$ converges iff $|r| < 1$, sum $\frac{a}{1-r}$ (first term over $1-r$)

p-series $\sum \frac{1}{n^p}$ converges iff $p > 1$; the harmonic series ($p = 1$) diverges

Telescoping partial fractions, write s_N, cancel, take $\lim s_N$

Integral $a_n = f(n)$, f positive, continuous, decreasing for $x \geq N$: $\sum a_n$ and $\int_N^{\infty} f\,dx$ share fate (not value)

Comparison, $0 \leq a_n \leq b_n$: $\sum b_n$ converges, so $\sum a_n$ does; $\sum a_n$ diverges, so $\sum b_n$ does

Limit comparison, $a_n, b_n > 0$: $\lim \frac{a_n}{b_n} = c$ with $0 < c < \infty$ means same fate

Alternating $b_n > 0$, decreasing, $b_n \to 0$, so $\sum (-1)^n b_n$ converges

Ratio $L = \lim \left| \frac{a_{n+1}}{a_n} \right|$; $L < 1$ converges absolutely, $L > 1$ diverges, $L = 1$ no conclusion

→ Factorials cancel: $\frac{(n+1)!}{n!} = n+1$ · $\frac{(2n+2)!}{(2n)!} = (2n+2)(2n+1)$

ABSOLUTE VS CONDITIONAL

→ Test $\sum |a_n|$ first: absolute convergence means convergence, done in one test.

e.g. $\sum \frac{(-1)^n}{n^2}$: $\sum \frac{1}{n^2}$ converges ($p = 2$), so absolutely

→ Say it in full: "By the ___ test (hypotheses: ___), the series converges / diverges (absolutely / conditionally)."

VECTORS, LINES & PLANES

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ · $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

Cross product

$$\vec{u} \times \vec{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

→ Middle term gets a minus. Check: the result dots to 0 with both.

e.g. $\langle 1,0,1 \rangle \times \langle 0,1,1 \rangle = \langle -1,-1,1 \rangle$

Vector projection $\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$ (an arrow) · scalar $\text{comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$ (a number)

Line through (x_0, y_0, z_0), direction $\langle a, b, c \rangle$: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$

Plane through (x_0, y_0, z_0), normal $\langle a, b, c \rangle$: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Distance from (x_1, y_1, z_1) to $ax + by + cz + d = 0$: $\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

→ Two lines: use different parameters (t and s); a solution of two equations must satisfy the third.

CONIC SECTIONS

→ Complete the square: factor the leading coefficient out of each group first; inside $9(\ldots)$, adding 4 really adds 36.

Ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, $a > b$ (a semi-major): $c^2 = a^2 - b^2$

e.g. $\frac{x^2}{25} + \frac{y^2}{9} = 1$: $c^2 = 25 - 9 = 16$, foci $(\pm 4, 0)$

Hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$: $c^2 = a^2 + b^2$, opens left and right, asymptotes $y - k = \pm \frac{b}{a}(x - h)$, directrices $x = h \pm \frac{a^2}{c}$

POWER & TAYLOR SERIES

Taylor at a: $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ · Maclaurin: $a = 0$

→ Interval of convergence: the ratio test gives the open interval; test each endpoint with its own named test.

$e^x = \sum \frac{x^n}{n!}$ (all x) · $\frac{1}{1-x} = \sum x^n$ for $|x| < 1$

e.g. Put $2x$ in for x:

$$e^{2x} = \sum \frac{(2x)^n}{n!} = 1 + 2x + 2x^2 + \cdots$$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ (all x)

WHICH MOVE?

If you see… → do this. First match wins.

- Something next to its derivative → **u-substitution**
- Root of a quadratic → **Trig substitution (match the sign pattern)**
- $\ln$ · power, $e^{\cdot}$ · trig, x · trig → **Parts (LIATE); $e^{\cdot}$ · trig cycles**
- Rational function → **Degree check, divide, partial fractions**
- Powers of $\sin$ and $\cos$ → **Odd: save one; both even: half-angle**
- Infinite bound or blow-up → **Swap the bad bound for t, integrate, then take the limit**
- Looks like $\frac{1}{n^p}$ → **p-series, comparison, limit comparison**
- Factorials or c^n → **Ratio test**
- $(-1)^n$ → **Absolute first, then alternating**
- $\sum c_n (x-a)^n$ → **Ratio test, then each endpoint**

⚠ COMMON SLIPS

Tick one when it bites you. Two ticks: slow down there.

- ☐ Forgetting $+C$ on an indefinite integral
- ☐ u-sub on a definite integral: limits not converted
- ☐ Polar area without the $\frac{1}{2}$, or bounds that trace twice
- ☐ $\frac{d^2y}{dx^2}$: not dividing by $\frac{dx}{dt}$ a second time
- ☐ Vector projection when the scalar was asked (or the reverse)
- ☐ $\int u^{-1/2}\,du = 2u^{1/2} + C$; check by differentiating back

QUICK CHECKS

Cover the sheet. Do these cold. Answers are upside down.

1. $\int xe^x\,dx$
2. $\int_1^{\infty} \frac{dx}{x^2}$
3. $\int \frac{dx}{x^2 - 1}$
4. $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$
5. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
6. Interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$
7. Area inside $r = 2\cos\theta$
8. Distance from the origin to $x + 2y + 2z = 1$

1. $(x-1)e^x + C$ **2.** 1 **3.** $\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$ **4.** 3 **5.** converges conditionally **6.** $[-1, 1)$ **7.** π **8.** $\frac{1}{3}$