ODU MATH SEQUENCE · CHEAT SHEETS

Precalc to Calc II

One double-sided landscape sheet per course, from MATH 162M through MATH 212.

MATH 162M
Precalculus I
the algebra toolkit
step 1 of 4 · pp. 3–4

→

MATH 163
Precalculus II
trig, vectors, conics
step 2 of 4 · pp. 5–6

→

MATH 211
Calculus I
limits, derivatives, integrals
step 3 of 4 · pp. 7–8

→

MATH 212
Calculus II
integration, series, polar, vectors
step 4 of 4 · pp. 9–10

Taking MATH 166 (Precalc I and II in one semester)? Use the 162M and 163 sheets together.
Topics follow the ODU 2026–27 catalog. Your instructor's syllabus always wins.

Edition 3 · 2026-10-18

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HOW TO USE A SHEET

1. New to it, or coming back after years? Learn the three numbered boxes first, in order. The first one is the words.
2. Read each box's **e.g.** line before its formulas: it is one solved example.
3. Cover one box, write what is in it from memory, then check.
4. Stuck? Read **Which move?** on the back before the formulas.
5. After homework, tick the **Common slips** you made. Two ticks: slow down there.
6. Keep the earlier sheets. Most calculus errors are algebra errors.

HOW TO PRINT

- US Letter, landscape, 100% scale, two-sided, flip on the **short** edge.
- Printed whole, each course lands on its own sheet.
- One course only? Print its two pages from the list on the front.

WHERE THE MATH COMES FROM

- Formulas are standard mathematics, checked line by line against OpenStax: *College Algebra 2e*, *Precalculus 2e*, and *Calculus* Volumes 1–3 (openstax.org).
- Course topics: the ODU 2026–27 catalog. Each sheet's footer says when to re-check it.
- Fact-checked independently on 2026-10-17, every quick-check answer worked by hand and by computer algebra. Edition 3's examples and word meanings were checked the same way on 2026-10-18.

CREDITS

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MATH 162M · Precalculus I

Start here 1 The words, in plain English 2 Algebraic operations 3 Equations *Study tool. Not for use on a test unless your instructor allows it.*

1 THE WORDS, IN PLAIN ENGLISH

coefficient	the number in front of a letter: in $5x^2$ it is 5
constant term	the number with no x attached
degree	the highest power of x
polynomial	numbers times whole-number powers of x, added up
domain	every x you are allowed to put in
zero (root)	an x that makes $f(x) = 0$
conjugate	same two terms, middle sign flipped: $3 + \sqrt{2}$ and $3 - \sqrt{2}$
discriminant	$b^2 - 4ac$, the part under the root; its sign tells you how many real answers
extraneous	an answer the algebra makes that fails the original
asymptote	a line the graph gets closer and closer to far out; it can cross a horizontal one
multiplicity	how many times a factor repeats: $(x-2)^3$ is 3
one-to-one	no two inputs give the same output
logarithm	the exponent: $\log_2 8 = 3$ because $2^3 = 8$

2 ALGEBRAIC OPERATIONS

$a^m a^n = a^{m+n} \cdot \dfrac{a^m}{a^n} = a^{m-n} \cdot (a^m)^n = a^{mn}$

$(ab)^n = a^n b^n \cdot a^0 = 1 \cdot a^{-n} = \dfrac{1}{a^n}\ (a \neq 0)$

$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$ for $a \geq 0$

$\sqrt{ab} = \sqrt{a}\,\sqrt{b} \cdot \sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$ for $a \geq 0, b > 0$

e.g. Rationalize:

$$\frac{1}{3+\sqrt{2}} \cdot \frac{3-\sqrt{2}}{3-\sqrt{2}} = \frac{3-\sqrt{2}}{7}$$

Complex numbers $i = \sqrt{-1}, i^2 = -1 \cdot (a+bi)(a-bi) = a^2 + b^2$

⚠ $\sqrt{x^2} = |x|$, not x

FACTORING & SPECIAL PRODUCTS

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$a^2 - b^2 = (a+b)(a-b)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

→ Order: pull out the greatest common factor (GCF), then count terms. **2 terms:** difference of squares, or sum or difference of cubes. **3 terms:** find two numbers that multiply to ac and add to b. **4 terms:** grouping.

e.g. $x^2 + 5x + 6$: $2 \cdot 3 = 6$ and $2 + 3 = 5$, so $(x+2)(x+3)$

⚠ $a^2 + b^2$ does not factor over the real numbers

RATIONAL EXPRESSIONS

→ Factor everything first, then cancel common **factors**.

Multiply $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$ · Divide: flip the second fraction, then multiply

Add or subtract rewrite over the least common denominator (LCD), then combine the numerators

→ Complex fraction: multiply top and bottom by the LCD of all the small fractions.

Excluded values anything that makes a denominator 0, including the top of a fraction you divide by

e.g. $\dfrac{x^2-9}{x+3} = \dfrac{(x+3)(x-3)}{x+3} = x - 3$, with $x \neq -3$

⚠ Cancel factors, never terms: $\dfrac{x+3}{3} \neq x + 1$

3 EQUATIONS

Quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

e.g. $x^2 + 2x - 3 = 0$: $b^2 - 4ac = 16$, so $x = \dfrac{-2 \pm 4}{2} = 1$ or -3

Discriminant $b^2 - 4ac$: positive, two real · zero, one real · negative, two complex

Complete the square $x^2 + bx = \left(x + \dfrac{b}{2}\right)^2 - \left(\dfrac{b}{2}\right)^2$

$|X| = k$ with $k \geq 0$: $X = k$ or $X = -k$ · $k < 0$: no solution

→ Radical: isolate the root, raise both sides to the power, solve, **check every answer**.

→ Rational: multiply by the LCD, solve, throw out excluded values.

→ Quadratic type: $x^4 - 5x^2 + 4 = 0$, let $u = x^2$

INEQUALITIES

⚠ Multiply or divide by a negative: flip the sign.

e.g. $-2x < 6$: divide by -2 and flip, $x > -3$

$|X| < k$ means $-k < X < k$

$|X| > k$ means $X < -k$ or $X > k$

→ Polynomial or rational: get 0 on one side, find the zeros and the undefined points, test one number in each interval (sign chart).

Undefined points are never included; zeros only for $<$ or $>$

Interval notation [ ] includes, ( ) excludes, $\pm\infty$ always takes ( )

GRAPHS & LINES

Distance $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Midpoint $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Circle $(x-h)^2 + (y-k)^2 = r^2$, center (h, k)

Slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ · point-slope $y - y_1 = m(x - x_1)$

$y = mx + b$ · parallel: same m · perpendicular: $m_2 = -\dfrac{1}{m_1}$ (horizontal ⊥ vertical)

e.g. Slope through $(1, 2)$ and $(3, 8)$: $\dfrac{8-2}{3-1} = 3$

→ Intercepts: for the x-intercept set $y = 0$; for the y-intercept set $x = 0$.

Symmetry x-axis if $(x, -y)$ also works · y-axis if $(-x, y)$ · origin if $(-x, -y)$

FUNCTIONS

Domain, rule out a denominator of 0 · an even root of a negative · a log of 0 or less

Composition $(f \circ g)(x) = f(g(x))$, inside first

e.g. $f(x) = x^2, g(x) = x + 1$: $f(g(2)) = f(3) = 9$

Difference quotient $\dfrac{f(x+h) - f(x)}{h}$ (becomes the derivative in Calc 1)

Average rate of change on $[a, b]$: $\dfrac{f(b) - f(a)}{b - a}$

Even $f(-x) = f(x)$ · Odd: $f(-x) = -f(x)$

→ Inverse (only if f is one-to-one): swap x and y, solve for y. Check $f(f^{-1}(x)) = x$ both ways.

One-to-one passes the horizontal line test · f^{-1} is f mirrored over $y = x$

⚠ $f^{-1}(x) \neq \dfrac{1}{f(x)}$

MATH 162M · Precalculus I

GRAPHS OF FUNCTIONS: TRANSFORMATIONS

$$y = a\,f\big(b(x-h)\big)+k$$

h: right $h\cdot$ k: up k

$|a|>1$ stretch · $0<|a|<1$ shrink · $a<0$ flip over the x-axis

b: horizontal scale by $\dfrac{1}{|b|}\cdot$ $b<0$ flip over the y-axis

e.g. Factor out b first: $f(2x+6)=f\big(2(x+3)\big)$ is a horizontal shrink by half and a shift left 3 (not 6).

⚠ $f(x+3)$ moves LEFT 3; $f(x)+3$ moves UP 3

POLYNOMIAL FUNCTIONS

Vertex form $f(x)=a(x-h)^2+k$, vertex (h,k)

From ax^2+bx+c: vertex at $x=-\dfrac{b}{2a}$, then $y=f\left(-\dfrac{b}{2a}\right)$

e.g. x^2-4x+1: $x=-\dfrac{-4}{2\cdot 1}=2$, $y=4-8+1=-3$, vertex $(2,-3)$

End behavior follows the leading term a_nx^n: even n, both ends go the way of the sign of a_n; odd n, the right end follows the sign of a_n and the left end goes the other way

Zero of odd multiplicity — the graph crosses · even multiplicity: touches and turns

Degree n: at most n real zeros and $n-1$ turning points

RATIONAL FUNCTIONS

Hole where a factor cancels completely from the bottom · vertical asymptote where the denominator is 0 after canceling

Horizontal asymptote, degree top n vs bottom m: $n<m, y=0$ · $n=m, y=\dfrac{\text{lead coef top}}{\text{lead coef bottom}}$ · $n>m$, none ($n=m+1$ gives a slant asymptote by long division)

e.g. $\dfrac{2x+1}{x-3}$: vertical $x=3$, horizontal $y=2$

THEORY OF EQUATIONS

Remainder theorem — $f(x)\div(x-c)$ leaves remainder $f(c)$

Factor theorem — $x-c$ is a factor exactly when $f(c)=0$

e.g. $f(x)=x^3-1$: $f(1)=0$, so $x-1$ is a factor

Rational zero candidates — $\pm\dfrac{p}{q}$, p divides the constant term, q divides the leading coefficient (integer coefficients)

→ Test candidates with synthetic division. Remainder 0 means a zero: keep going on the smaller quotient.

Descartes — positive zeros = sign changes in $f(x)$, or fewer by an even number; negative zeros: count in $f(-x)$

Fundamental theorem of algebra: degree n has exactly n complex zeros, counting multiplicity

Real coefficients — complex zeros come in pairs $a\pm bi$

SYSTEMS OF EQUATIONS

→ Substitution: solve one equation for a variable, plug it into the other.

→ Elimination: scale the equations so one variable cancels when you add.

e.g. $y=2x$ and $x+y=6$: $x+2x=6$, so $x=2, y=4$

One solution · none ($0=5$, parallel lines) · infinitely many ($0=0$, same line)

Row operations — swap rows · multiply a row by a nonzero number · add a multiple of one row to another

$$\det\begin{bmatrix}a & b\\ c & d\end{bmatrix}=ad-bc$$

Nonlinear system — substitute, then check each pair in both equations

EXPONENTIAL FUNCTIONS

$f(x)=a\cdot b^x$ with $a>0$: $b>1$ growth · $0<b<1$ decay · passes $(0,a)$ · asymptote $y=0$

Compound interest — $A=P\left(1+\dfrac{r}{n}\right)^{nt}$

Continuous — $A=Pe^{rt}\cdot e\approx 2.71828$

Growth or decay — $A=A_0e^{kt}$ ($k>0$ grows, $k<0$ decays)

e.g. Same base, set the exponents equal: $2^{x+1}=2^5$ gives $x=4$.

LOGARITHMIC FUNCTIONS

$y=\log_b x$ means $b^y=x$, for $b>0, b\neq 1$, $x>0$ (the log IS the exponent)

e.g. $\log_3 x=2$ means $x=3^2=9$

$\log_b 1=0\cdot\log_b b=1\cdot\log_b b^x=x\cdot b^{\log_b x}=x$ $(x>0)$

Product — $\log_b(MN)=\log_b M+\log_b N$ $(M,N>0)$

Quotient — $\log_b\dfrac{M}{N}=\log_b M-\log_b N$ ($M,N>0$)

Power — $\log_b M^p=p\log_b M$ for $M>0$

Change of base — $\log_b x=\dfrac{\ln x}{\ln b}\cdot$ ln is $\log_e$, log is $\log_{10}$

→ Exponential equation: isolate the power, take ln of both sides.

→ Log equation: condense to one log, rewrite as an exponential, **check x is in the domain**.

⚠ $\log(a+b)\neq\log a+\log b\cdot\dfrac{\log a}{\log b}\neq\log\dfrac{a}{b}$

WHICH MOVE?

If you see… → do this. First match wins.

- x^2 and no plain x term → **Isolate x^2, take $\pm\sqrt{}$**
- Quadratic → **Factor if it is quick; otherwise the formula**
- Degree 3 or more → **Rational zero candidates, synthetic division**
- x in a denominator → **Multiply by the LCD; check excluded values**
- x under a root → **Isolate the root, raise to the power, check**
- x in an exponent → **Same base: equal exponents; else take ln**
- x inside a log → **Condense, rewrite as exponential, check**
- Absolute value → **Split into two cases**
- Inequality with x^2 or a fraction → **Zeros and undefined points, sign chart**

⚠ COMMON SLIPS

Tick one when it bites you. Two ticks: slow down there.

- ☐ $(a+b)^2\neq a^2+b^2$ (multiply it out)
- ☐ $\sqrt{a^2+b^2}\neq a+b$
- ☐ $-3^2=-9$, but $(-3)^2=9$
- ☐ Distribute the minus: $-(x-4)=-x+4$
- ☐ Use $+$ when you square-root both sides

QUICK CHECKS

Cover the sheet. Do these cold. Answers are upside down.

1. Solve $x^2-6x+5=0$
2. Solve $|2x-1|<5$
3. Domain of $f(x)=\dfrac{\sqrt{x-3}}{x-5}$
4. Inverse of $f(x)=2x+7$
5. Solve $3^{x+1}=81$
6. Solve $\log_2 x+\log_2(x-2)=3$
7. Remainder of x^3-2x+4 divided by $x-2$
8. Solve $x+y=5$ and $x-y=1$

1. $x=1,\ 5$ **2.** $-2<x<3$ **3.** $[3,5)\cup(5,\infty)$ **4.** $f^{-1}(x)=\dfrac{x-7}{2}$ **5.** $x=3$ **6.** $x=4$ ($x=-2$ rejected) **7.** 8 **8.** $(3,\ 2)$

MATH 163 · Precalculus II

Start here 1 The words, in plain English 2 Trigonometric functions: unit circle 3 Trigonometric identities

Study tool. Not for use on a test unless your instructor allows it.

1 THE WORDS, IN PLAIN ENGLISH

radian	angle measured by arc length on a radius-1 circle: π rad $= 180^\circ$
coterminal	angles that end on the same ray: 30° and 390°
reference angle	the acute angle between the angle's ending side and the x-axis
quadrant	one of the four quarters of the plane, I to IV counterclockwise
amplitude	half the height from lowest to highest point
period	how far along x before the graph repeats
phase shift	how far the graph slides left or right
midline	the horizontal line halfway between top and bottom
identity	an equation true for every allowed x
arcsine	the angle in $[-\pi/2, \pi/2]$ whose sine is given: $\sin^{-1}(1/2) = \pi/6$
magnitude	a vector's length
orthogonal	at right angles; for vectors, dot product 0
modulus, argument	a complex number's distance from 0, and its angle
focus	a fixed point (two for an ellipse or hyperbola) the conic is built around

EXPONENTIAL & LOGARITHMIC FUNCTIONS (FROM 162M)

$y = \log_b x$ means $b^y = x$ ($b > 0, b \neq 1, x > 0$) · ln is $\log_e$

$\log_b(MN) = \log_b M + \log_b N$ · $\log_b \dfrac{M}{N} = \log_b M - \log_b N$ $(M, N > 0)$

→ Exponential equation: isolate, take ln. Log equation: condense, rewrite as exponential, check the domain.

e.g. $e^{2x} = 5$: $2x = \ln 5$, so $x = \dfrac{\ln 5}{2} \approx 0.805$

ANGLES

π rad $= 180^\circ$ · degrees to radians $\times \dfrac{\pi}{180}$ · radians to degrees $\times \dfrac{180}{\pi}$

Arc length $s = r\theta$ · sector area $A = \dfrac{1}{2}r^2\theta$ (θ in radians)

e.g. $r = 4, \theta = 150^\circ = \dfrac{5\pi}{6}$: $s = 4 \cdot \dfrac{5\pi}{6} = \dfrac{10\pi}{3}$

Linear speed $v = r\omega$ (ω in rad/time) · coterminal: add or subtract 2π (360°)

Reference angle (acute, to the x-axis; first bring θ into $[0, 2\pi)$; QI–QIV are quadrants I–IV, counterclockwise from upper right): QII $\pi - \theta$ · QIII $\theta - \pi$ · QIV $2\pi - \theta$

TRIGONOMETRIC FUNCTIONS: RIGHT TRIANGLES

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$ · $\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$ · $\tan\theta = \dfrac{\text{opp}}{\text{adj}}$ (SOH CAH TOA)

$\csc\theta = \dfrac{1}{\sin\theta}$ · $\sec\theta = \dfrac{1}{\cos\theta}$ · $\cot\theta = \dfrac{1}{\tan\theta}$

e.g. Opposite 3, hypotenuse 5: adjacent 4, so $\sin\theta = \dfrac{3}{5}, \cos\theta = \dfrac{4}{5}, \tan\theta = \dfrac{3}{4}$

30-60-90 sides $1 : \sqrt{3} : 2$ (opposite $30^\circ, 60^\circ, 90^\circ$) · 45-45-90 sides $1 : 1 : \sqrt{2}$

2 TRIGONOMETRIC FUNCTIONS: UNIT CIRCLE

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
	0°	30°	45°	60°	90°
$\sin$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef

A point on the unit circle is $(\cos\theta, \sin\theta)$ · $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Positive QI all · QII sin · QIII tan · QIV cos, and their reciprocals ("All Students Take Calculus")

→ Exact value anywhere: reference angle from the table, sign from the quadrant.

e.g. $\sin\dfrac{5\pi}{6}$: reference angle $\dfrac{\pi}{6}$, QII so positive, $\dfrac{1}{2}$

GRAPHS OF TRIGONOMETRIC FUNCTIONS

$y = A\sin\big(B(x - C)\big) + D$

Amplitude $|A|$ · period $\dfrac{2\pi}{|B|}$ · phase shift C · midline $y = D$

e.g. $y = 2\sin 3x + 1$: amplitude 2, period $\dfrac{2\pi}{3}$, midline $y = 1$

tan and cot: period $\dfrac{\pi}{|B|}$ · $y = \tan x$ has asymptotes at $x = \dfrac{\pi}{2} + k\pi$

sin, cos: range $[-1, 1]$ · domain all real numbers

→ Sketch one period in four equal steps: five key points.

⚠ Factor out B: $\sin(2x - \pi) = \sin\big(2(x - \dfrac{\pi}{2})\big)$, the shift is $\dfrac{\pi}{2}$, not π

INVERSE TRIGONOMETRIC FUNCTIONS

$\sin^{-1} x$ returns an angle in $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$\cos^{-1} x$ returns $[0, \pi]$ · $\tan^{-1} x$ returns $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

e.g. $\cos^{-1}\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$, the angle in $[0, \pi]$

⚠ $\sin^{-1} x$ means arcsine, not $\dfrac{1}{\sin x}$

3 TRIGONOMETRIC IDENTITIES

$\sin^2 x + \cos^2 x = 1$ · $1 + \tan^2 x = \sec^2 x$ · $1 + \cot^2 x = \csc^2 x$

$\cos(-x) = \cos x$ · $\sin(-x) = -\sin x$ · $\tan(-x) = -\tan x$

Cofunction $\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$

$\sin(a \pm b) = \sin a\cos b \pm \cos a \sin b$

$\cos(a \pm b) = \cos a\cos b \mp \sin a \sin b$

e.g. $\sin 75^\circ = \sin(45^\circ + 30^\circ) = \dfrac{\sqrt{6} + \sqrt{2}}{4}$

$\tan(a \pm b) = \dfrac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$

$\sin 2x = 2\sin x\cos x$ · $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

$\cos 2x = \cos^2 x - \sin^2 x$, also $2\cos^2 x - 1$ or $1 - 2\sin^2 x$

Power-reducing $\sin^2 x = \dfrac{1 - \cos 2x}{2}$ · $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

Half-angle $\sin\dfrac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{2}}$ · $\cos\dfrac{x}{2} = \pm\sqrt{\dfrac{1 + \cos x}{2}}$, sign from the quadrant of $\dfrac{x}{2}$

→ Proving one: work on the messier side only, rewrite in sin and cos, combine fractions, factor, look for $\sin^2 + \cos^2$.

LAW OF SINES & LAW OF COSINES

Given parts, in order around the triangle: S = side, A = angle. SAS = two sides and the angle between them; SSA = two sides and an angle not between them.

Sines $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, for AAS, ASA, SSA

e.g. $A = 30^\circ, B = 45^\circ, a = 10$: $b = \dfrac{10 \sin 45^\circ}{\sin 30^\circ} = 10\sqrt{2}$

Cosines $c^2 = a^2 + b^2 - 2ab\cos C$, for SAS, SSS

Area $= \dfrac{1}{2}ab\sin C$ · Heron: $\sqrt{s(s-a)(s-b)(s-c)}$, $s = \dfrac{a+b+c}{2}$

→ SSS: find the largest angle first; the other two are then acute.

⚠ SSA can give 0, 1, or 2 triangles: if B works, also try $180^\circ - B$; keep it only if $A + (180^\circ - B) < 180^\circ$.

VECTORS

$\mathbf{v} = \langle a, b\rangle \cdot |\mathbf{v}| = \sqrt{a^2+b^2}$ · unit vector $\dfrac{\mathbf{v}}{|\mathbf{v}|}$

From length and angle $\mathbf{v} = \langle |\mathbf{v}|\cos\theta,\ |\mathbf{v}|\sin\theta\rangle$

Direction $\tan\theta = \dfrac{b}{a}$, then fix the quadrant

Dot product $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2$ · $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$

Perpendicular (orthogonal) exactly when $\mathbf{u}\cdot\mathbf{v} = 0$

e.g. $\langle 1, 2\rangle \cdot \langle 4, -2\rangle = 4 - 4 = 0$, so they are orthogonal

POLAR FORM OF COMPLEX NUMBERS

$z = a + bi = r(\cos\theta + i\sin\theta) \cdot r = \sqrt{a^2+b^2}$ · $\tan\theta = \dfrac{b}{a}$ (check the quadrant)

e.g. $1 + i$: $r = \sqrt{2}, \theta = \dfrac{\pi}{4}$, so $\sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

Multiply multiply the r's, add the angles · Divide: divide the r's, subtract the angles

De Moivre $z^n = r^n(\cos n\theta + i\sin n\theta)$

nth roots: $\sqrt[n]{r}\left(\cos\dfrac{\theta+2k\pi}{n} + i\sin\dfrac{\theta+2k\pi}{n}\right)$, $k = 0, 1, \ldots, n-1$

Polar points $x = r\cos\theta \cdot y = r\sin\theta$ · $r^2 = x^2 + y^2$

BINOMIAL THEOREM

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

$\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ = entry k of row n in Pascal's triangle (count both from 0)

e.g. $\binom{6}{2} = \dfrac{6!}{2!\,4!} = 15$

The term with b^k (term number $k+1$): $\binom{n}{k}a^{n-k}b^k$

⚠ Keep the sign inside b: in $(x-2)^5$, $b = -2$.

TRIGONOMETRIC EQUATIONS

→ **1** Isolate the trig function. **2** Find the reference angle. **3** List every angle in $[0, 2\pi)$ with the right sign. **4** General solution: add $2k\pi$ (for $\tan$ and $\cot$, $k\pi$).

→ Two different trig functions: use an identity to get one, or factor.

e.g. $\sin 2x = \dfrac{\sqrt{2}}{2}$ on $[0, 2\pi)$: $2x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{9\pi}{4}, \dfrac{11\pi}{4}$ on $[0, 4\pi)$, so $x = \dfrac{\pi}{8}, \dfrac{3\pi}{8}, \dfrac{9\pi}{8}, \dfrac{11\pi}{8}$

⚠ Never divide both sides by $\sin x$: factor it out, or you lose the $\sin x = 0$ answers.

CONIC SECTIONS

Parabola $(x-h)^2 = 4p(y-k)$: vertex (h, k), focus $(h, k+p)$, opens up if $p > 0$ · $(y-k)^2 = 4p(x-h)$ opens right if $p > 0$

Ellipse $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, long axis horizontal; for vertical, a^2 goes under $(y-k)^2$. Always $a > b$ and $c^2 = a^2 - b^2$

Hyperbola $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$, $c^2 = a^2 + b^2$, asymptotes $y - k = \pm\dfrac{b}{a}(x-h)$; $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$ opens up and down, slopes $\pm\dfrac{a}{b}$ (a^2 always under the positive term)

e.g. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$: $a = 5, b = 3$, $c = \sqrt{25-9} = 4$, foci $(\pm 4, 0)$

→ Identify $Ax^2 + Cy^2 + Dx + Ey + F = 0$: $A = C \neq 0$ circle · $AC > 0$ ellipse · $AC < 0$ hyperbola · one squared term parabola. Then complete the square: a 0 or negative on the right can mean a point, two lines, or no graph.

WHICH MOVE?

If you see… → do this. First match wins.

- Right triangle → **SOH CAH TOA**
- SSS or SAS → **Law of Cosines**
- AAS or ASA → **Law of Sines**
- SSA → **Law of Sines, then look for a second triangle**
- Only $2x$ inside the trig function → **Solve for $2x$ on the doubled interval, then divide**
- $2x$ mixed with x, or two trig functions → **Double-angle identity, or factor**
- Power or root of a complex number → **Polar form, De Moivre**
- One term of $(a+b)^n$ → **Term formula, not the full expansion**

⚠ COMMON SLIPS

Tick one when it bites you. Two ticks: slow down there.

- ☐ $\sin 2x \neq 2\sin x$ and $\sin(a+b) \neq \sin a + \sin b$
- ☐ Degree mode or radian mode: match the problem
- ☐ Arc length and sector area need radians
- ☐ Find every solution in the interval, not just the first
- ☐ After $\tan^{-1}$, check the quadrant

QUICK CHECKS

Cover the sheet. Do these cold. Answers are upside down.

1. $\cos\dfrac{2\pi}{3}$
2. Solve $2\sin x - 1 = 0$ on $[0, 2\pi)$
3. Period of $y = 3\cos 4x$
4. $|\langle 3, -4\rangle|$
5. $(1+i)^8$
6. Coefficient of x^2 in $(x+2)^5$
7. $a = 7,\ b = 8,\ C = 60^\circ$. Find c.
8. Which conic is $x^2 + 4y^2 = 16$?

1. $-\dfrac{1}{2}$ 2. $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ 3. $\dfrac{\pi}{2}$ 4. 5 5. 16 6. 80

7. $c = \sqrt{57}$ 8. ellipse, $a = 4,\ b = 2$

MATH 211 · Calculus I

Start here 1 The words, in plain English 2 Limits 3 Differentiation rules *Study tool. Not for use on a test unless your instructor allows it.*

1 THE WORDS, IN PLAIN ENGLISH

limit	the value $f(x)$ heads toward as x gets close to a
continuous	no break, hole or jump: $\lim_{x\to a} f(x) = f(a)$
indeterminate form	a form like $0/0$ or ∞/∞ (also $0\cdot\infty, \infty-\infty, 1^\infty, 0^0, \infty^0$): not an answer, a sign to do more work
derivative	the slope of the graph at one point; the instant rate of change
differentiable	has a derivative there: smooth, no corner, no break, no vertical tangent
tangent line	the line through the point with the same slope as the graph there
implicit	y is mixed into the equation, not solved for
critical number	an x in the domain where $f'=0$ or f' does not exist
extrema	highest and lowest values (maximum and minimum)
concave up	bends upward like a cup; $f''>0$ is the usual test
inflection point	where the bend changes direction
antiderivative	a function whose derivative is the one you have
definite integral	area between the graph and the x-axis from a to b; area below the axis counts negative
integrand	the function inside the integral sign

PRECALC YOU WILL USE EVERY DAY

$\sqrt{x} = x^{1/2}\cdot\dfrac{1}{x^n} = x^{-n}\cdot\dfrac{1}{\sqrt{x}} = x^{-1/2}$ (rewrite before differentiating)

e.g. $\dfrac{1}{\sqrt{x}} = x^{-1/2}$, so its derivative is $-\dfrac{1}{2}x^{-3/2}$

→ Most Calc 1 mistakes are algebra mistakes. Keep the 162M and 163 sheets.

2 LIMITS

$\lim_{x\to c} f(x)$ exists only when the left and right limits exist and are equal

→ **1** Plug in. A number, and f is continuous there? Done. At a piecewise break, take one-sided limits.

→ **2** $\dfrac{\text{nonzero}}{0}$: vertical asymptote. Check each side for $+\infty$ or $-\infty$.

→ **3** $\dfrac{0}{0}$: factor and cancel, multiply by the conjugate, or clear small fractions. Or L'Hôpital.

e.g. $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2}(x+2) = 4$

$\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1\cdot\lim\limits_{x\to 0}\dfrac{1-\cos x}{x} = 0$

$x\to+\infty$: divide top and bottom by the highest power of x in the denominator

Squeeze $g\le f\le h$ and $\lim g = \lim h = L$, so $\lim f = L$

CONTINUITY

Continuous at c: $f(c)$ is defined, $\lim_{x\to c} f(x)$ exists, and the two are equal

e.g. $\dfrac{x^2-1}{x-1}$ is not continuous at $x=1$: it is undefined there

Intermediate Value Theorem: continuous on $[a,b]$ means f takes every value between $f(a)$ and $f(b)$

THE DERIVATIVE

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

e.g. $f(x)=x^2$: $\dfrac{(x+h)^2-x^2}{h} = 2x+h\to 2x$

Tangent line at $x=a$: $y-f(a) = f'(a)(x-a)$

f' is the slope of the tangent, the instantaneous rate of change

Not differentiable at a corner, a cusp, a vertical tangent, or a break

3 DIFFERENTIATION RULES

$(c)' = 0\cdot(x^n)' = nx^{n-1}\cdot(cf)' = cf'\cdot(f\pm g)' = f'\pm g'$

Product $(fg)' = f'g + fg'$

Quotient $\left(\dfrac{f}{g}\right)' = \dfrac{f'g-fg'}{g^2}$

Chain $\big[f(g(x))\big]' = f'(g(x))\cdot g'(x)$

→ Chain rule in words: derivative of the outside (inside left alone), times the derivative of the inside.

e.g. $\big[(x^2+1)^5\big]' = 5(x^2+1)^4\cdot 2x$

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS

$(\sin x)' = \cos x\cdot(\cos x)' = -\sin x\cdot(\tan x)' = \sec^2 x$

$(\cot x)' = -\csc^2 x\cdot(\sec x)' = \sec x\tan x\cdot(\csc x)' = -\csc x\cot x$

$(e^x)' = e^x\cdot(a^x)' = a^x\ln a\ (a>0)$

$(\ln x)' = \dfrac{1}{x}\cdot(\log_a x)' = \dfrac{1}{x\ln a}\ (a>0,\ a\ne 1)$

$(\sin^{-1}x)' = \dfrac{1}{\sqrt{1-x^2}}\cdot$

$(\tan^{-1}x)' = \dfrac{1}{1+x^2}$

e.g. $(e^x\sin x)' = e^x\sin x + e^x\cos x$ (product rule)

→ The "co" functions (cos, cot, csc) carry the minus sign.

IMPLICIT DIFFERENTIATION

→ Differentiate both sides in x. Every time you differentiate a y, attach y'. Then solve for y'

e.g. $x^2+y^2=25$ gives $2x+2yy'=0$, so $y' = -\dfrac{x}{y}$

Variable in base and exponent (x^x): take ln of both sides first

APPLICATIONS: RELATED RATES & LINEAR APPROXIMATION

→ **1** Draw and label. **2** Write an equation linking the quantities. **3** Differentiate with respect to t. **4** THEN plug in the numbers.

e.g. $A=\pi r^2$ gives $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$. With $r=3$ and $\dfrac{dr}{dt}=2$, the area grows at 12π

Linear approximation $L(x) = f(a)+f'(a)(x-a)$ · differential $dy = f'(x)\,dx$

APPLICATIONS: EXTREMA & SHAPE

Critical number — c inside the domain with $f'(c) = 0$ or $f'(c)$ undefined

$f' > 0$ increasing · $f' < 0$ decreasing · at a critical number where f is continuous, f' from $+$ to $-$ is a local max, $-$ to $+$ a local min

$f'' > 0$ concave up · $f'' < 0$ concave down · inflection point: f continuous there and concavity changes

e.g. $f(x) = x^2 - 4x$: $f' = 2x - 4 = 0$ at $x = 2$, $f'' = 2 > 0$, local min $f(2) = -4$

Second derivative test — $f'(c) = 0$ with $f''(c) < 0$ is a local max; with $f''(c) > 0$ a local min; $f''(c) = 0$ tells you nothing, use the f' sign chart

→ Closed interval $[a, b]$, f continuous: evaluate f at every critical number AND both endpoints. Largest is the absolute max, smallest the absolute min.

Mean Value Theorem — if f is continuous on $[a, b]$ and differentiable on (a, b), then $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some c in (a, b)

APPLICATIONS: OPTIMIZATION

→ **1** Draw. **2** Write the quantity to maximize or minimize. **3** Use the constraint to get one variable. **4** State the domain. **5** Find the critical numbers: $f' = 0$ or f' undefined. **6** Compare them with the endpoints, or use the sign of f'. **7** Answer the question asked, with units.

e.g. Two numbers add to 10; largest product: $P = x(10 - x)$, $P' = 10 - 2x = 0, x = 5, P = 25$

L'HÔPITAL'S RULE

$\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$: $\lim \dfrac{f(x)}{g(x)} = \lim \dfrac{f'(x)}{g'(x)}$ if f and g are differentiable near the point, $g' \neq 0$ there, and the new limit exists or is $\pm\infty$

⚠ Differentiate top and bottom separately, not with the quotient rule.

e.g. $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = \lim\limits_{x \to 0} \dfrac{e^x}{1} = 1$

$0 \cdot \infty$ or $\infty - \infty$: rewrite as one fraction first

$1^\infty, 0^0, \infty^0$: take $\ln$, find that limit, then raise e to it

INTEGRATION: ANTIDERIVATIVES

$\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C$ for $n \neq -1$

$\int \dfrac{1}{x} \, dx = \ln|x| + C$ · $\int e^x \, dx = e^x + C$

e.g. $\int \left(3x^2 + \dfrac{1}{x}\right) dx = x^3 + \ln|x| + C$

$\int \cos x \, dx = \sin x + C$ ·
$\int \sin x \, dx = -\cos x + C$ ·
$\int \sec^2 x \, dx = \tan x + C$ ·
$\int \sec x \tan x \, dx = \sec x + C$ ·
$\int \dfrac{dx}{1 + x^2} = \tan^{-1} x + C$ ·
$\int \dfrac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C$

$\int a^x \, dx = \dfrac{a^x}{\ln a} + C$ for $a > 0, a \neq 1$

INTEGRATION: U-SUBSTITUTION

→ Pick u = the inside function whose derivative (up to a constant) is also there. Write $du = u'(x) \, dx$.

e.g. $\int xe^{x^2} \, dx$: $u = x^2$, $du = 2x \, dx$, answer $\dfrac{1}{2}e^{x^2} + C$

⚠ Definite integral: change the bounds to u-values, or switch back to x before plugging in.

INTEGRATION: DEFINITE INTEGRALS & THE FTC

Riemann sum — $\sum f(x_i^*) \, \Delta x$ with $\Delta x = \dfrac{b - a}{n}$

Fundamental Theorem of Calculus (FTC), part 1: f continuous, $\dfrac{d}{dx} \int_a^x f(t) \, dt = f(x)$

With a chain:

$$\frac{d}{dx} \int_a^{g(x)} f(t) \, dt = f(g(x)) \, g'(x)$$

FTC part 2 — f continuous on $[a, b]$, $F' = f$: $\int_a^b f(x) \, dx = F(b) - F(a)$

e.g. $\int_1^3 2x \, dx = 3^2 - 1^2 = 8$

$\int_a^a f = 0$ · $\int_a^b f = -\int_a^b f$

Average value — $\dfrac{1}{b - a} \int_a^b f(x) \, dx$ · area between curves $\int_a^b (\text{top} - \text{bottom}) \, dx$, split where the curves cross

WHICH MOVE?

If you see… → do this. First match wins.

- Any limit → **Plug in first**
- $\dfrac{0}{0}$ → **Factor, conjugate, or L'Hôpital**
- Product of two functions → **Product rule**
- A fraction → **Quotient rule, or rewrite with a negative exponent**
- Function inside a function → **Chain rule**
- x and y mixed together → **Implicit differentiation**
- "How fast is it changing" → **Related rates**
- "Largest, smallest, most" → **Optimization**
- Integral on the table → **Reverse the table, add C**
- An inside function and its derivative → **u-substitution**
- $\dfrac{d}{dx} \int_a^{g(x)}$ → **FTC part 1 and the chain rule**

⚠ COMMON SLIPS

Tick one when it bites you. Two ticks: slow down there.

- ☐ Missing the inner derivative in the chain rule
- ☐ Quotient rule: $f'g - fg'$ on top (order matters), g^2 below
- ☐ Forgetting $+C$
- ☐ $\int \dfrac{1}{x} \, dx = \ln|x| + C$, not the power rule
- ☐ $\dfrac{d}{dx}(e^2) = 0$, it is a constant

QUICK CHECKS

Cover the sheet. Do these cold. Answers are upside down.

1. $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$
2. $\dfrac{d}{dx}\left[x^2 \sin x\right]$
3. $\dfrac{d}{dx}\left[\ln(x^2 + 1)\right]$
4. $\dfrac{d}{dx}\left[e^{3x}\right]$
5. $\int_0^2 3x^2 \, dx$
6. $\int 2x \cos(x^2) \, dx$
7. $\lim\limits_{x \to \infty} \dfrac{3x^2 + 1}{5x^2 - x}$
8. Critical numbers of $f(x) = x^3 - 3x$

8. $x = \mp 1$

4. $3e^{3x}$ 5. 8 6. $\sin(x^2) + C$ 7. $\dfrac{3}{5}$

1. 6 2. $2x \sin x + x^2 \cos x$ 3. $\dfrac{2x}{x^2 + 1}$

MATH 212 · Calculus II

Start here 1 The words, in plain English 2 Integration by parts 3 Infinite series: tests

Study tool. Not for use on a test unless your instructor allows it.

1 THE WORDS, IN PLAIN ENGLISH

integrand	the function inside the integral sign
irreducible	a quadratic that will not factor over the reals: x^2+1
improper integral	an infinite bound, or a blow-up inside: done as a limit
converges	settles on one finite number
diverges	does not settle: grows without bound or keeps bouncing
sequence, series	a list $a_1, a_2, \ldots$; a series adds the list up
partial sum	S_n, the sum of the first n terms
absolutely convergent	still converges with every term made positive
conditionally convergent	converges, but not with every term made positive
radius of convergence	how far from the center it converges; test each endpoint alone
Taylor series	a function rebuilt as an infinite polynomial around $x=a$
parametric	x and y each given by a third variable t
polar	a point given by distance r from the origin and angle θ
normal vector	a vector at right angles to a plane

TRIGONOMETRIC SUBSTITUTION

$\sqrt{a^2-x^2}$: $x=a\sin\theta$, root $=a\cos\theta$

$\sqrt{a^2+x^2}$: $x=a\tan\theta$, root $=a\sec\theta$

$\sqrt{x^2-a^2}$: $x=a\sec\theta$, root $=a|\tan\theta|$, which is $a\tan\theta$ when $x\geq a$

→ Back to x: draw the right triangle the substitution describes.

e.g. $\displaystyle\int\frac{dx}{\sqrt{4-x^2}}$: $x=2\sin\theta$, $dx=2\cos\theta\,d\theta$, root $=2\cos\theta$, so $\displaystyle\int d\theta=\sin^{-1}\frac{x}{2}+C$

⚠ $\sqrt{\cos^2\theta}=|\cos\theta|$: drop the bars only once you show it is positive

2 INTEGRATION BY PARTS

$$\int u\,dv=uv-\int v\,du$$

→ Pick u by LIATE: Log, Inverse trig, Algebraic, Trig, Exponential. First on the list is u.

e.g. $\int x\cos x\,dx$: $u=x, v=\sin x$, so $x\sin x+\cos x+C$

Definite:

$$\int_a^b u\,dv=\big[uv\big]_a^b-\int_a^b v\,du$$

→ Cycle ($e^{ax}\sin bx$ or $e^{ax}\cos bx$): parts twice, same kind of u both rounds. The integral I comes back; move it left and solve for I.

$$\int e^{ax}\sin bx\,dx=\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}+C$$

TRIGONOMETRIC INTEGRALS

Odd power of sin: save one $\sin x$, rest to $1-\cos^2 x$, $u=\cos x$ ($du=-\sin x\,dx$)

e.g. $\int\sin^3 x\,dx$ with $u=\cos x$: $-\cos x+\dfrac{\cos^3 x}{3}+C$

Odd power of cos: save one $\cos x$, rest to $1-\sin^2 x$, $u=\sin x$

Both even $\quad\sin^2 x=\dfrac{1-\cos 2x}{2}\cdot\quad\cos^2 x=\dfrac{1+\cos 2x}{2}$

$\int\tan x\,dx=\ln|\sec x|+C\cdot$

$\int\sec x\,dx=\ln|\sec x+\tan x|+C$

$\int\csc x\,dx=\ln|\csc x-\cot x|+C$

PARTIAL FRACTIONS

→ Degree check first: top degree $\geq$ bottom degree means long division before splitting.

Distinct linear ($a\neq b$):

$$\frac{px+q}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}$$

Repeated $\quad\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}\cdot$

irreducible quadratic: $\dfrac{Ax+B}{x^2+bx+c}$

→ Cover-up: clear denominators, set $x=a$, then $x=b$; one constant drops out each time.

e.g. $\dfrac{1}{x(x+1)}=\dfrac{A}{x}+\dfrac{B}{x+1}$: $x=0$ gives $A=1$, $x=-1$ gives $B=-1$

$\displaystyle\int\frac{dx}{ax+b}=\frac{1}{a}\ln|ax+b|+C$, not the power rule

IMPROPER INTEGRALS

Infinite bound $\quad\displaystyle\int_a^\infty f\,dx=\lim_{t\to\infty}\int_a^t f\,dx$

Blow-up at b: $\displaystyle\lim_{t\to b^-}\int_a^t f\,dx\cdot$ at a: $\displaystyle\lim_{t\to a^+}\int_t^b f\,dx$

Blow-up inside, at c: split at c; it converges only if both pieces converge

e.g. $\displaystyle\int_0^1\frac{dx}{\sqrt{x}}=\lim_{t\to0^+}\big[2\sqrt{x}\big]_t^1=2$

p-tests $\quad\displaystyle\int_1^\infty\frac{dx}{x^p}$ converges iff $p>1\cdot$

$\displaystyle\int_0^1\frac{dx}{x^p}$ converges iff $p<1$

⚠ Scan for hidden blow-ups ($\ln x$ at 0, a zero denominator) before integrating.

POLAR COORDINATES

$x=r\cos\theta\cdot y=r\sin\theta\cdot r^2=x^2+y^2\cdot\tan\theta=\dfrac{y}{x}$

e.g. $r=2,\theta=\dfrac{\pi}{3}$: $x=2\cos\dfrac{\pi}{3}=1$, $y=2\sin\dfrac{\pi}{3}=\sqrt{3}$

Area $\quad A=\dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2\,d\theta\cdot$ between curves $\dfrac{1}{2}\displaystyle\int_\alpha^\beta\left(r_{\text{out}}^2-r_{\text{in}}^2\right)d\theta$

Arc length $\quad L=\displaystyle\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

→ Bounds trace the region once: $r=a\cos\theta$ closes on $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$; 0 to 2π counts it twice.

→ Intersections: set the r's equal, then check the pole separately.

Limaçon $r=a+b\sin\theta$ (or $\cos\theta$), $a,b>0$: inner loop when $a<b$; the loop lies between the θ's where $r=0$

⚠ Negative r plots opposite θ.

PARAMETRIC CURVES

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}\cdot\quad\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\Big/\frac{dx}{dt}$$

e.g. $x=t^2, y=t^3$: $\dfrac{dy}{dx}=\dfrac{3t^2}{2t}=\dfrac{3t}{2}$ $(t\neq0)$

Horizontal tangent $\quad\dfrac{dy}{dt}=0$ and $\dfrac{dx}{dt}\neq0\cdot$ vertical: $\dfrac{dx}{dt}=0$ and $\dfrac{dy}{dt}\neq0$

Arc length $L=\displaystyle\int_\alpha^\beta ds$, where

$$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Surface area $\quad S=2\pi\displaystyle\int_\alpha^\beta(\text{radius})\,ds$; radius is y about the x-axis, x about the y-axis (radius >0, curve traced once)

MATH 212 · Calculus II

3 INFINITE SERIES: TESTS

Divergence $\lim a_n \neq 0$ means $\sum a_n$ diverges

⚠ $a_n \to 0$ alone proves nothing.

e.g. $\sum \dfrac{n}{2n+1}$: the terms go to $\dfrac{1}{2} \neq 0$, so it diverges

Geometric $\sum_{n=0}^{\infty} ar^n$ converges iff $|r| < 1$, sum $\dfrac{a}{1-r}$ (first term over $1-r$)

p-series $\sum \dfrac{1}{n^p}$ converges iff $p > 1$; the harmonic series ($p = 1$) diverges

Telescoping partial fractions, write s_N, cancel, take $\lim s_N$

Integral $a_n = f(n)$, f positive, continuous, decreasing for $x \geq N$: $\sum a_n$ and $\int_N^{\infty} f\,dx$ share fate (not value)

Comparison, $0 \leq a_n \leq b_n$: $\sum b_n$ converges, so $\sum a_n$ does; $\sum a_n$ diverges, so $\sum b_n$ does

Limit comparison, $a_n, b_n > 0$: $\lim \dfrac{a_n}{b_n} = c$ with $0 < c < \infty$ means same fate

Alternating $b_n > 0$, decreasing, $b_n \to 0$, so $\sum (-1)^n b_n$ converges

Ratio $L = \lim \left|\dfrac{a_{n+1}}{a_n}\right|$; $L < 1$ converges absolutely, $L > 1$ diverges, $L = 1$ no conclusion

→ Factorials cancel: $\dfrac{(n+1)!}{n!} = n+1$ · $\dfrac{(2n+2)!}{(2n)!} = (2n+2)(2n+1)$

ABSOLUTE VS CONDITIONAL

→ Test $\sum |a_n|$ first: absolute convergence means convergence, done in one test.

e.g. $\sum \dfrac{(-1)^n}{n^2}$: $\sum \dfrac{1}{n^2}$ converges ($p = 2$), so absolutely

→ Say it in full: "By the ___ test (hypotheses: ___), the series converges / diverges (absolutely / conditionally)."

VECTORS, LINES & PLANES

$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$ · $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$

Cross product

$$\vec{u} \times \vec{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

→ Middle term gets a minus. Check: the result dots to 0 with both.

e.g. $\langle 1,0,1\rangle \times \langle 0,1,1\rangle = \langle -1,-1,1\rangle$

Vector projection $\text{proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v}$ (an arrow) · scalar $\text{comp}_{\vec{v}}\vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$ (a number)

Line through (x_0, y_0, z_0), direction $\langle a, b, c\rangle$: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$

Plane through (x_0, y_0, z_0), normal $\langle a, b, c\rangle$: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Distance from (x_1, y_1, z_1) to $ax + by + cz + d = 0$: $\dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

→ Two lines: use different parameters (t and s); a solution of two equations must satisfy the third.

CONIC SECTIONS

→ Complete the square: factor the leading coefficient out of each group first; inside $9(\ldots)$, adding 4 really adds 36.

Ellipse $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, $a > b$ (a semi-major): $c^2 = a^2 - b^2$

e.g. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$: $c^2 = 25 - 9 = 16$, foci $(\pm 4, 0)$

Hyperbola $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$: $c^2 = a^2 + b^2$, opens left and right, asymptotes $y - k = \pm\dfrac{b}{a}(x - h)$, directrices $x = h \pm \dfrac{a^2}{c}$

POWER & TAYLOR SERIES

Taylor at a: $\sum_{k=0}^{\infty} \dfrac{f^{(k)}(a)}{k!}(x-a)^k$ · Maclaurin: $a = 0$

→ Interval of convergence: the ratio test gives the open interval; test each endpoint with its own named test.

$e^x = \sum \dfrac{x^n}{n!}$ (all x) · $\dfrac{1}{1-x} = \sum x^n$ for $|x| < 1$

e.g. Put $2x$ in for x:

$$e^{2x} = \sum \frac{(2x)^n}{n!} = 1 + 2x + 2x^2 + \cdots$$

$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$

$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots$ (all x)

WHICH MOVE?

If you see… → do this. First match wins.

- Something next to its derivative → **u-substitution**
- Root of a quadratic → **Trig substitution (match the sign pattern)**
- $\ln$ · power, $e^{\cdot}$ · trig, x · trig → **Parts (LIATE); $e^{\cdot}$ · trig cycles**
- Rational function → **Degree check, divide, partial fractions**
- Powers of $\sin$ and $\cos$ → **Odd: save one; both even: half-angle**
- Infinite bound or blow-up → **Swap the bad bound for t, integrate, then take the limit**
- Looks like $\dfrac{1}{n^p}$ → **p-series, comparison, limit comparison**
- Factorials or c^n → **Ratio test**
- $(-1)^n$ → **Absolute first, then alternating**
- $\sum c_n(x-a)^n$ → **Ratio test, then each endpoint**

⚠ COMMON SLIPS

Tick one when it bites you. Two ticks: slow down there.

- ☐ Forgetting $+C$ on an indefinite integral
- ☐ u-sub on a definite integral: limits not converted
- ☐ Polar area without the $\dfrac{1}{2}$, or bounds that trace twice
- ☐ $\dfrac{d^2y}{dx^2}$: not dividing by $\dfrac{dx}{dt}$ a second time
- ☐ Vector projection when the scalar was asked (or the reverse)
- ☐ $\int u^{-1/2}\,du = 2u^{1/2} + C$; check by differentiating back

QUICK CHECKS

Cover the sheet. Do these cold. Answers are upside down.

1. $\int xe^x\,dx$
2. $\int_1^{\infty} \dfrac{dx}{x^2}$
3. $\int \dfrac{dx}{x^2 - 1}$
4. $\sum_{n=0}^{\infty} \left(\dfrac{2}{3}\right)^n$
5. $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n}$
6. Interval of convergence of $\sum_{n=1}^{\infty} \dfrac{x^n}{n}$
7. Area inside $r = 2\cos\theta$
8. Distance from the origin to $x + 2y + 2z = 1$

8. $\frac{1}{3}$ · 7. π · 6. $[-1, 1)$ · 5. converges conditionally · 4. 3 · 3. $\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$ · 2. 1 · 1. $(x-1)e^x + C$